ČETRTI LETNIK — 1994–1995 – 2

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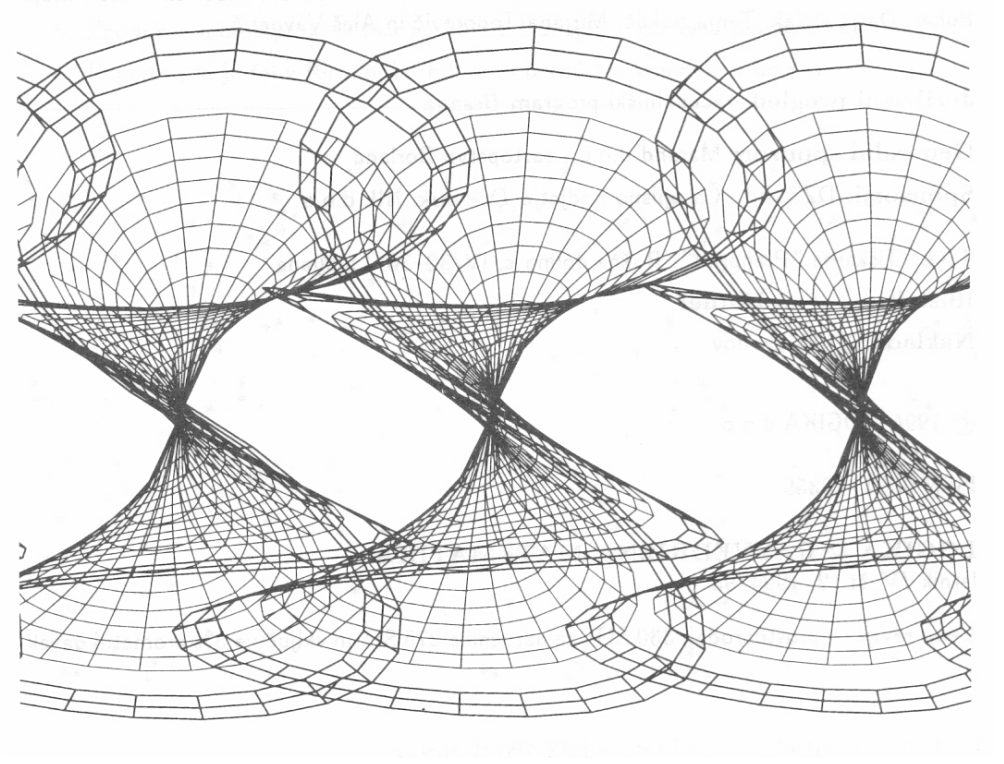
&

RAZVEDRILNA MATEMATIKA

Revijo sta za splet pripravila Nada in Marko Razpet
na podlagi datotek, ki jih je izdelal Darjo Felda.

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International Mathenmatical Talent Search 27



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Sodelavci: Urša Demšar, Gregor Dolinar, Urška Drčar, Petra Ipavec, Alenka Kavčič, Dušanka Kocić, Katka Kurent, Meta Lah, Nina Milač, Nika Novak, Hiacinta Pintar, Maja Pohar, Darja Polak, Tanja Soklič, Mirjana Todorovič in Aleš Vavpetič

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Cena revije: v prosti prodaji 330 SIT, za naročnike 275 SIT in vključuje 5% prometni davek

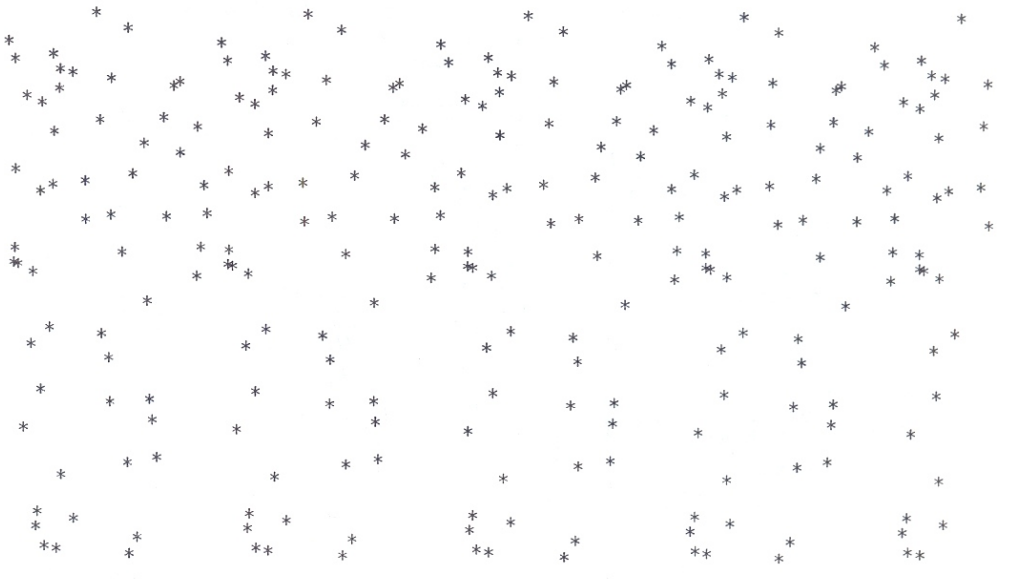
Logične naloge

1. NOVINARJI

Vodja radia "Kravja vas"je poslal pet novinarjev na lov za razburljivimi zgodbami v različna velika mesta. Ugotovi, kateri reporter je šel v določen kraj, kakšno zgodbo je našel in kdo je bil vpleten v zgodbi!

Novinarji: Ana, Edi, Hugo, Peter, Sandra;
Kraji: Rim, Amsterdam, Atene, Dunaj, Pariz;
Zgodbe: pijan voznik, škandal v bolnici, osramočenje, rop, umor;
Osebe v zgodbah: knezova hči, okrožni sodnik, mornar, pevka, župnik.

1. Zgodba o pevki iz Pariza je bila na radijskih valovih, preden je Edi sporočil na radio zgodbo o umoru.
2. Okrožnemu sodniku so ukradli zbirko srebrnih tobačnic.
3. Nekdo je bil osramočen v Amsterdamu. Knezova hči, katere zgodbo je obravnaval Peter, pri osramočenju ni bila vpletena.
4. Ana je šla v Atene.
5. Zgodbe o mornarju, ki ni bila poslana z Dunaja, ni obravnavala Sandra.
6. Reporter, ki je šel v Rim, ni bil Edi in ni poročal o škandalu v bolnici.



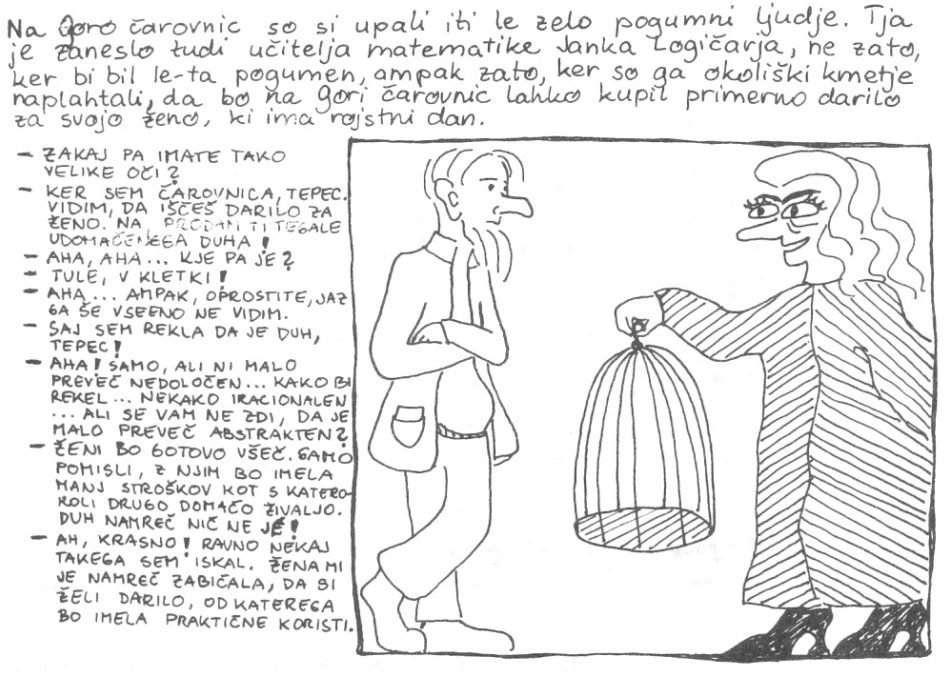
Borut Jurčič Zlobec

2. ČAROVNICE

Globoko v osrčju gozda stoji velika skalna gmota, ki ji tamkajšnji kmetje pravijo Gora čarovnic in se je izognejo, kadar le morejo. Včasih pa le pridejo tja, da obiščejo eno od čarovnic, ki živijo vsaka v svoji jami. Ugotovi polno ime vsake čarovnice, za kaj je izurjena in v kateri jami prebiva.

Imena: Agata, Hilda, Gizela, Hilarija, Regina;
Vzdevki: Temačna, Črnogleda, Zlobna, Senčna, Volčja;
Specializacije: klicanje duhov, prekletstvo, zdravila, napoj za srečo, strupi.

1. Strupi so bili pripravljeni v jami, ki je takoj nad tisto, v kateri je živela Črnogleda Hilda.
2. Regina, katere vzdevek ni bil Volčja, je klicala duhove; ni živela v jami št. 1.
3. Gizela je živela v jami št. 4.
4. Napoje za srečo je pripravljala čarovnica v jami št. 2, ki pa ni bila Hilarija.
5. Čarovnica, ki je izdelovala zdravila, je imela vzdevek Senčna.
6. Agata se je izogibala jame št. 2, ker je vlažnost slabšala njen revmatizem; njena lastna jama ni bila takoj pod jamo Volčje.
7. Čarovnica Zlobna je živela v jami takoj pod tisto jamo, v kateri je druga čarovnica izrekala prekletstva.

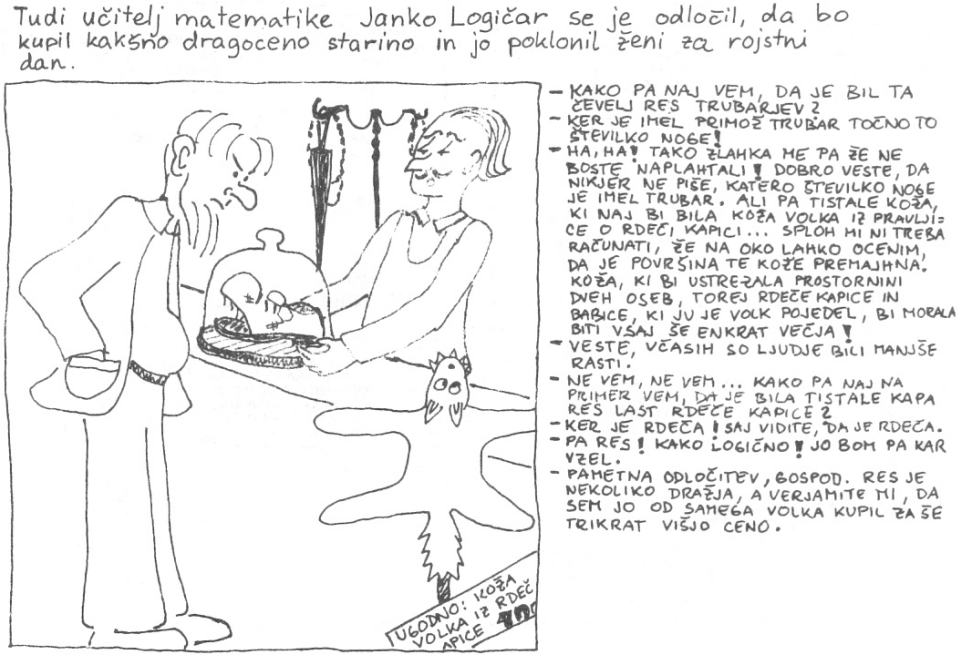


3. ZBIRALEC STARIN

Znanec, ki zbira starine, mi je pred kratkim pripovedoval o ugodnih, a sumljivih kupčijah, ki jih je sklenil. Iz spodnjih trditev ugotovi kraj in ime trgovine, kjer je sklenil kupčijo, dragoceni predmet in domnevnega slavnega lastnika.

Trgovine: Antikvariat, Podstrešje, Zastavljalnica, Stari časi, Spomini;
Kraji: Maribor, Ptuj, Velenje, Novo mesto, Celje;
Predmeti: čaša, obesek, mošnja, prstan, čevelj;
Osebnosti: Turjaška Rozamunda, cesarica Marija Terezija, Martin Krpan, Primož Trubar, kralj Matjaž.

1. Predmet, ki ga je zbiralec odkril v zakotni mariborski trgovinici, naj bi bil osebna lastnina kralja Matjaža.
2. Prstan je iz trgovine Stari časi.
3. Dragocenost, ki naj bi jo svoje čase uporabljala cesarica Marija Terezija, ni iz trgovine Zastavljalnica.
4. Zlata čaša iz velenjske trgovine, ki pa se ne imenuje Spomini, ni bila last Turjaške Rozamunde.
5. Ne predmet, kupljen v Antikvariatu na Ptuju, niti srebrni obesek nista bila last katere od žensk.
6. Čevelj Primoža Trubarja je bil kupljen v kraju z daljšim imenom, kot ga ima mesto s trgovino Spomini.



4. ČRNI PETEK

V soboto, štirinajstega, se je pet prijateljev pogovarjalo o nesrečah, ki so se jim zgodile dan poprej (torej v petek, trinajstega). Ugotovi, kakšne nezgode so se jim pripetile, kakšno predhodno znamenje jim je nesrečo napovedalo in kaj so po poklicu.

Osebe: Ana, Karla, Janez, Robert, Samo;
Poklici: voznik avtobusa, blagajnik, prodajalec zelenjave, tajnik, voznik tovornjaka;
Znamenja: razbil zrcalo, čudne sanje, črna mačka, raztresel sol, kolcanje;
Nesreče: izgubil uro, zamudil letalo, izgubil službo, bil oropan, izpahnil gleženj.

1. Najmlajši izmed prijateljev si je že zjutraj izpahnil gleženj, potem ko je imel čudne sanje.
2. Voznik tovornjaka ni bil tisti, ki se mu je kolcalo.
3. Janez, najstarejši izmed nesrečnikov, je po poklicu voznik. On ni raztresel soli niti ni zamudil letala.
4. Blagajniku je spodrsnilo na milu, tako da je razbil ogledalo v kopalnici. To je krivil za svojo poznejšo nesrečo.
5. Prodajalec zelenjave, ki ni Karla, je bil oropan.
6. Ana, ki si ni izpahnila gležnja, je tajnica. Robert je bil tisti, ki mu je črna mačka prekrižala pot.
7. Samo, ki po poklicu ni voznik, je v petek izgubil službo.



Natečaj za najboljšo logično nalogo

1. Zdraviliški turizem

Poletje, čas, ko se nihče ne more upreti toplemu soncu ali vsaj naravi, ki se mu ponuja. Takrat si vsak zasluži kratek počitek ob obali ali v planinah. Pet parov pa se je odločilo drugače. Izbrali so si toplice. S pomočjo vodilnih niti ugotovi imena fantov, imena deklet, kraj letovanja in kako dolgo so letovali.

Vodilne niti

1. Andrej ni obiskal toplic za pet dni, teden ali dva, njegova sopotnica pa je bila Olga.
2. Par, ki je obiskal Blejske toplice za štirinajst dni, je poslal razglednico prijateljema, ki sta preživela pet lepih dni, a ne v Čateških toplicah.
3. Neža in njen mož imata v bližini Šmarjeških toplic nečaka, katerega sta obiskala med desetdnevnim letovanjem.
4. Samo in Majda sta se z dopusta vrnila s poškodovanim avtom, vendar sta se veselila pozdravov z Bleda.
5. Polona, ki je bila v toplicah tri dni manj kot Peter, je oddih preživela z izbrancem, ki je prvič letoval v Čatežu.
6. Luka je letoval najdlje, soba v Dolenjskih toplicah pa je bila zasedena pet dni.

	Neža	Olga	Polona	Romana	Majda	Samo	Luka	Peter	Andrej	Tomaž	5 dni	7 dni	10 dni	12 dni	14 dni
Dolenjske t.															
Šmarješke t.															
Radenci															
Blejske t.															
Čateške t.															
5 dni															
7 dni															
10 dni															
12 dni															
14 dni															
Samo															
Luka															
Peter															
Andrej															
Tomaž															

2. Nagajivi Franček

Saj poznate tistega Frančka iz sedmega razreda, Frančka, katerega so zanimala dekleta vseh vrst. Do sedaj se ni še čisto nič spremenil, le bolj hudomušen je postal. Na nesrečo so bile žrtve njegovih šal sošolke. Iz podatkov ugotovi: ime, priimek, šalo in dan, na katerega jo je Franček zakuhal. Seveda so se dekleta med seboj sprla zaradi posmehovanja druge drugi, a so se kmalu spet pobotala in že kujejo načrt, kaj bodo one ušpičile nagajivemu sošolcu.

Vodilne niti

1. Tjaše, ki se ne piše Korošec, v ponedeljek ni bilo v šoli, zato tudi ni imela na koncu šolske ure skupaj zavezanih vezalk na čevljih.
2. Časova in Urška se bosta oddolžili za zvečilni gumi v laseh in za črnilo na stolu, a ne nujno v tem vrstnem redu, Pinterjeva pa je jezo že malo ohladila.
3. Medvedova Vida je Frančka ovadila razredničarki za žabo v njeni torbi, on pa jo je skrivaj pobrisal.
4. Novakovo je Franček dražil v petek, Vido pa v sredo.
5. Lucija ni prišla v sredo v šolo zaradi žvečilnega gumija, ki ji je skrajšal dolžino las za petnajst centimetrov.
6. Dekle, ki se piše Novak, se je hotela po uri telesne vzgoje osvežiti s kozarcem mrzle vode, a voda ni bila samo mrzla ampak tudi slana.

	Pinter	Korošec	Medved	Novak	Časa	pon.	torek	sreda	četrtek	petek	Tjaša	Lucija	Vida	Eva	Urška
črnilo															
vezalke															
žaba															
sol															
žvečilka															
Tjaša															
Lucija															
Vida															
Eva															
Urška															
ponedeljek															
torek															
sreda															
četrtek															
petek															

3. Ulična svetilka

Park je kraj, kamor prihajajo zaljubljeni ljudje. Podnevi se v krošnjah košatih dreves poigravajo sončni žarki, ponoči pa zelenju daje čar svetloba neonskih svetilk. Ena izmed teh stoji na koncu, kjer park meji na ulico, pa ima posebno vlogo. ob njej se zbira pet parov. Iz naslednjih podatkov s pomočjo razpredelnice določi, ob katerem času se je kateri par sestal in kam se je namenil preživeti večer.

Vodilne niti:

1. Par, ki je bil povabljen na rojstni dan, se je sestal pol ure pred Anžetom in Vido, ki pa tudi nista prišla najkasneje.
2. Šolska zabava se je začela uro prej, preden sta se na sprehod odpravila Nežka in njen fant, ki pa ni Tadej.
3. Par, ki si je ogledal film, nista Robert in Alja, ki se tudi nista udeležila šolske zabave.
4. Blaž in Mojca sta brat in sestra. Med njunima odhodoma na zmenek so minile štriri ure.
5. Par se je odpravil v disko ob pol devetih.
6. Tadej si ni ogledal filma.

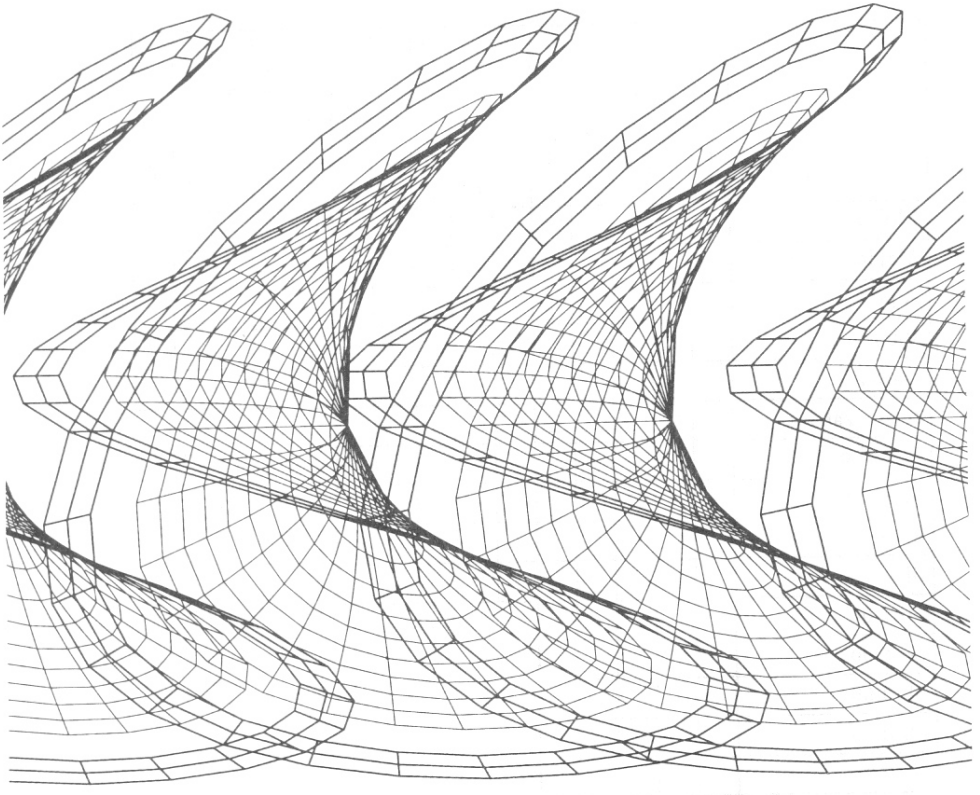
	Anže	Robert	Matija	Tadej	Blaž	ob 5^h	ob 6^h	ob 8^h	ob 8^{30}	ob 9^h	šol. zabava	rojst. dan	sprehod	kino	disko
Mojca															
Majda															
Alja															
Vida															
Nežka															
šolska zabava															
rojstni dan															
sprehod															
kino															
disko															
ob 5:00															
ob 6:00															
ob 8:00															
ob 8:30															
ob 9:00															

4. Zaroka

V štirih zaporednih dneh so izšle štiri nove številke časopisov, v katerih so bile objavljene zaroke. S pomočjo vodilnih niti določi: imena bodočih zakonskih parov, priimke, časopis in dan, na katerega je izšel.

Vodilne niti

1. Naš čas izide vsak četrtek, v njem pa je objavljena zaroka para, katerega priimka se začneta na isto črko, začetnici imen pa sta črki k in s.
2. Roman Nemec in Andreja sta se odločila za 7D, dan po zaroki pa sta njuno objavo že lahko prebrala v časopisu, ki sta si ga izbrala.
3. Zaroka gdč. Nataše in gospoda Kranjca ni bila objavljena v torek.
4. Kajin zaročenec ni gospod Dvor.
5. Gospodična Lenart in njen zaročenec sta objavila zaroko v časopisu, ki ne izide v sredo ali dan prej.
6. Vsi pari so se zaročili v nedeljo, 2. avgusta.
7. Natašin brat Aleš Urh bo poročna priča sestrinemu zaročencu Stanku.
8. Jana izide vsakih štirinajst dni; nova številka pa izide 18. avgusta.



Borut Jurčič Zlobec

	7 D	Stop	Jana	Naščas	Urh	Fele	Drcar	Lenart	Andreja	Saša	Kaja	Nataša	Kmetec	Nemec	Dvor	Kranjc	Stanko	Klemen	Lovro	Roman
ponedeljek																				
torek																				
sreda																				
četrtek																				
Stanko																				
Klemen																				
Lovro																				
Roman																				
Kmetec																				
Nemec																				
Dvor																				
Kranjc																				
Andreja																				
Saša																				
Kaja																				
Nataša																				
Urh																				
Fele																				
Drcar																				
Lenart																				

5. Srebrni svečniki

Zgodi se, da na letališču pride do zamenjave kovčkov, ali pa le-ti ne prispejo, kamor bi morali. Pet lastnikov pomotoma zamenjanih kovčkov je v direktorjevi pisarni pričakal inšpektor v spremstvu dveh stražnikov. Potnik, ki je nezakonito prekupčeval s srebrnimi svečniki, je ugotovil, da so odkrili dragocenosti. Vse njegove izjave so neresnične. Inšpektorjeva naloga je odkriti prekupčevalca in na podlagi pravilnih izjav vrniti stvari v kovčkih ostalim štirim lastnikom.

Anita: Prinesla sem siv kovček, no, tudi moj je takšne barve. V mojem kovčku so poleg malenkosti knjige za branje ob prostem času.

Vasja: Nekdo je pomotoma vzel moj zelen kovček.

Duška: Moj kovček je majhen. V kovčku, ki sem ga prinesla nazaj, so karte, v mojem pa je budilka.

Borut: Moj kovček ni sive barve, v katerem je budilka, ampak je rjav in v njem so knjige.

Grega: Moj kovček je velik in rjav, klobuka pa letos nisem vzel. Svečniki niso v majhnem kovčku.

	Anita	Vasja	Duška	Borut	Grega	karte	knjige	klobuk	budilka	svečniki
majhen siv										
velik siv										
velik rjav										
velik zelen										
maj. zelen										
karte										
knjige										
klobuk										
budilka										
svečniki										

Rešitve nalog

1. Zdraviliški turizem

ime fanta	ime dekleta	kraj letovanja	število dni
Samo	Majda	Dolenjske toplice	5 dni
Tomaž	Polona	Čateške toplice	7 dni
Peter	Neža	Šmarješke toplice	10 dni
Andrej	Olga	Radenci	12 dni
Luka	Romana	Blejske toplice	14 dni

2. Nagajivi Franček

ime	priimek	šala	dan
Eva	Pinter	zvezane vezalke	ponedeljek
Lucija	Čas	žvečilni gumi v laseh	torek
Vida	Medved	žaba v torbi	sreda
Urška	Korošec	črnilo na stolu	četrtek
Tjaša	Novak	soljena voda	petek

3. Ulična svetilka

Mojca in Tadej	šolska zabava	ob peti uri
Nežka in Matija	sprehod	ob šestih
Alja in Robert	rojstni dan	ob osmih
Vida in Anže	disko	ob pol devetih
Majda in Blaž	kino	ob devetih

4. Zaroka

ime in priimek	ime in priimek	časopis	dan izdaje
Roman Nemec	Andreja Lenart	7D	ponedeljek
Lovro Kmetec	Kaja Fele	Jana	torek
Stanko Kranec	Nataša Urh	Stop	sreda
Klemen Dvor	Saša Drcar	Naš čas	četrtek

5. Srebrni svečniki

Borut je prekupčevalec, saj se njegova izjava ne ujema z izjavami ostalih.

Anita	majhen siv kovček	knjige
Vasja	velik zelen kovček	klobuk
Duška	majhen zelen kovček	budilka
Borut	velik siv kovček	svečniki
Grega	velik rjav kovček	karte

Špela Janežič

Naloge za najmlajše

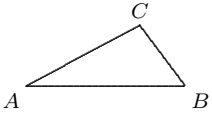
Učenci šestih razredov *OŠ Janka Glazerja* v Rušah smo se pri dodatnem pouku matematike in pri krožku logike preizkusili v sestavljanju logičnih nalog, križank...
Nekaj nalog pošiljamo tudi vam. Upamo, da se boste ob reševanju le-teh zabavali vsaj toliko, kot smo se mi ob njihovem nastajanju.

Lepo vas pozdravljamo šestošolci s svojo mentorico Leonido Osojnik.

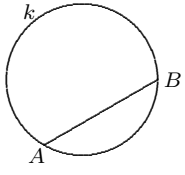
1. Križanka (*Adil Čehič*, 6. a)

Vodoravno:

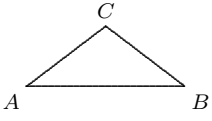
1. _____ do matematike.
2. Osnosomerni štirikotnik, v katerem gre somernica skozi nasprotni oglišči.
3. Množico točk, ki se preko premice prezrcali sama vase, imenujemo OSNO_____.
4. Poimenuj množico točk:



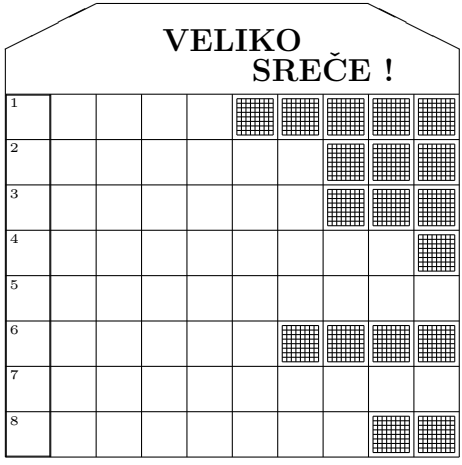
5. $80 - 54 = 26$ Kako imenujemo to računsko operacijo?
6. Poimenuj daljico AB.



7. To je _____ trikotnik.



8. $35 : 5 = 7$ Kako imenujemo rezultat pri deljenju?



Pod *1 navpično* dobiš rešitev križanke. Naštej vsaj tri področja, kjer boš to rešitev potreboval tudi ti!

2. Zimske igre (*Zora Zidarič*, 6. c)

Maja, Peter, Barbara, Tina in Rado so sodelovali v različnih zimskih igrah (sankanje, smučanje, kepanje, drsanje in postavljanje snežaka). Ugotovi, kdo se je udeležil vsake od iger, če je vsak sodeloval le v eni.

Veš tudi: – Maja ima sani, vendar jih je posodila Radu.
– Barbarina mama se jezi, ker ne najde lonca za juho, niti korenja.
– Tina ne mara drsanja.
– Peter je postal prvak v smučanju.

3. Športne dejavnosti (*Ajiz Čehič*, 6. a)

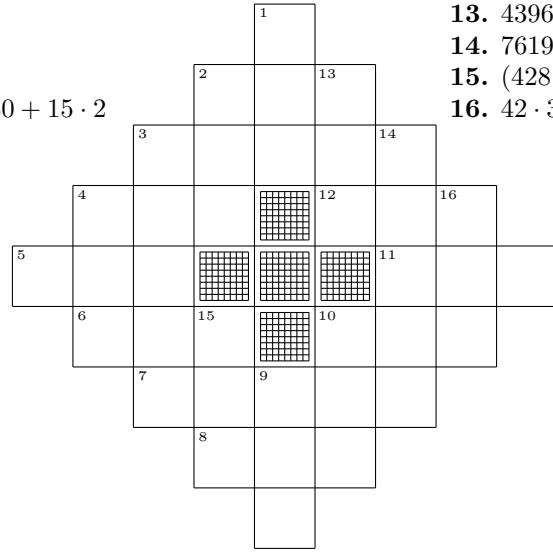
Marko, Peter, Sonja in Nataša obiskujejo različne športne dejavnosti: košarko, odbojko, namizni tenis in strelstvo. Ali veš, kdo obiskuje naštete dejavnosti, če vsak med njimi obiskuje le eno?

1. Marko ima rad šport, pri katerem rezultat ni odvisen od igre cele ekipe.
2. Mali Peter nima loparja.
3. Nataša ne potrebuje niti žoge niti loparja.

4. Križanka (*Miha Eder*, 6. b)

Vodoravno:
2. $3 \cdot 141$
3. $31239 + 22222 + 851$
4. $378 + 611$
5. $1600 - 4945 : 5$
6. $4 + 30 + 350 + 350$
7. $(50000 + 9) + (2000 + 40) + 800$
8. $8^3 + 3^2 + 3 \cdot 10^2 + 11$
10. $600 - 2 - 2 + 2 - 2$
11. 10% od 3210
12. $451 - 20 + 15 + 5 - 30 + 15 \cdot 2$

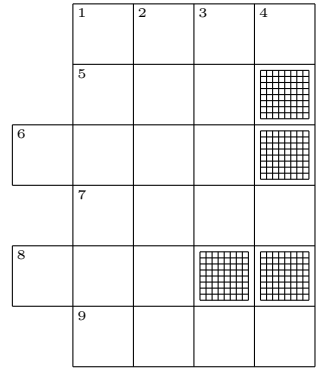
Navpično:
1. $321 - 198$
2. $400 + 49$
3. $58134 + 1$
4. $(917 : 2) \cdot 2$
9. $535 + 299$
10. $501 + 41$
13. $4396 : 14$
14. $76197 : 3$
15. $(428 : 2) \cdot (4 : 2)$
16. $42 \cdot 3$



5. Križanka (*Saša Baranašič*, 6. a)

Vodoravno:
1. $555 \cdot \frac{16}{5}$
5. $\frac{25}{3} \cdot 111$
6. $1894 - 800$
7. $855 \cdot \frac{29}{5}$
8. $313 \cdot 3$
9. $2043 + 365$

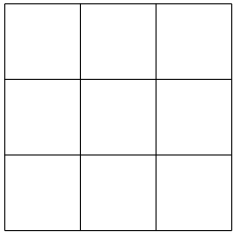
Navpično:
1. $190535 - 103$
2. $182973 + 547021$
3. $4263 + 3282$
4. sestavljeno število



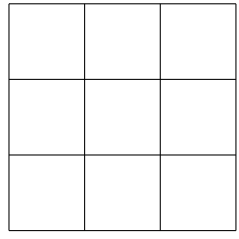
6. Anti-magični kvadrati (*Aleksandra Kosi*, 6. c)

Pravimo, da je kvadrat *anti-magični*, če so vsote števil, ki so postavljena po stolpcih, vrsticah in diagonalah, med seboj različne. Pri posamezni nalogi postavi v polja kvadrata predpisana števila tako, da bo dobljeni kvadrat anti-magični.

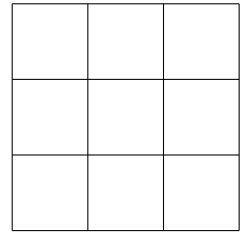
a) Vpiši števila od 12 do 20



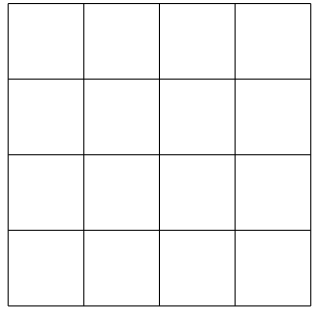
b) Vpiši števila od 2 do 10



c) Vpiši števila od 3 do 11



d) Vpiši števila od 1 do 16



Angleške naloge za srednješolce

Problems

1. Compute the sum of the first n odd numbers.
2. The positive integer n, when divided by 3, 4, 5, 6, and 7, leaves remainders of 2, 3, 4, 5, and 6, respectively. Find the least possible value of n.
3. Quadrilateral $ABCD$ has right angles at B and D. If $ABCD$ is kite shaped with $AB = AD = 20$ and $BC = CD = 15$, find the length of a radius of the circle inscribed in $ABCD$.
4. Find the coordinates of that point on the circle with equation $(x-6)^2+(y-5)^2 = 25$ that is nearest the point $(-2, 11)$.
5. If $a < b$, find the ordered pair of positive integers (a, b) that satisfies
$$\sqrt{10+\sqrt{84}} = \sqrt{a}+\sqrt{b}.$$
6. Find the value of a if the graphs of $2y + x + 3 = 0$ and $3y + ax + 2 = 0$ are to meet at right angles.
7. Let the roots of $ax^2 + bx + c = 0$ be r and s. Find the equation with roots $ar + b$ and $as + b$.
8. Given two equiangular polygons P_1 and P_2 with different numbers of sides; each angle of P_1 is x digrees and each angle of P_2 is kx degrees, where k is an integer greater than 1. What is the number of possibilities for the pair (x, k)?
9. What is the area of the region in the Cartesian plane whose points (x, y) satisfy $|x| + |y| + |x + y| \leq 2$?

Solutions

1. Denote the required sum by S_n, i.e.
$$S_n = 1 + 2 + 3 + ... + (2n - 1).$$
We give n the succesion of values 1, 2, 3,... until we have accumulated enough material upon which we can construct a more or less reliable hypothesis. After this, it only remains to check this hypothesis by method of mathematical induction.

We have $S_1 = 1$, $S_2 = 4$, $S_3 = 9$, $S_4 = 16$, $S_5 = 25$. Now everything depends on the problem-solver's keenness of observation, on his ability to guess a general result on the basis of particular results.

We think it is easy to notice in the case under consideration that $S_1 = 1^2$, $S_2 = 2^2$, $S_3 = 3^2$, $S_4 = 4^2$, $S_5 = 5^2$.

On the basis of this we may assume that in general
$$S_n = n^2$$
We shall prove that this hypothesis is valid.

a) For $n = 1$ the sum is a single summand equal to 1. The expression n^2 is also equal to 1. Hence the hypothesis is true for $n = 1$.

b) Assume that the hypothesis is true for $n = k$, i.e. $S_k = k^2$. Let us prove that hypothesis

must then also be true for $n = k + 1$, i.e.

$$S_{k+1} = (k+1)^2$$

In fact,

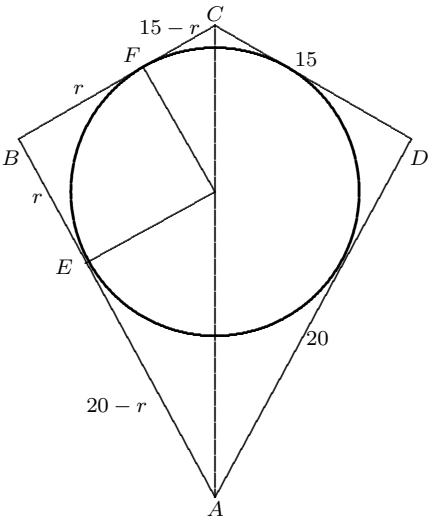
$$S_{k+1} = S_k + (2k+1)$$

But $S_k = k^2$ and therefore

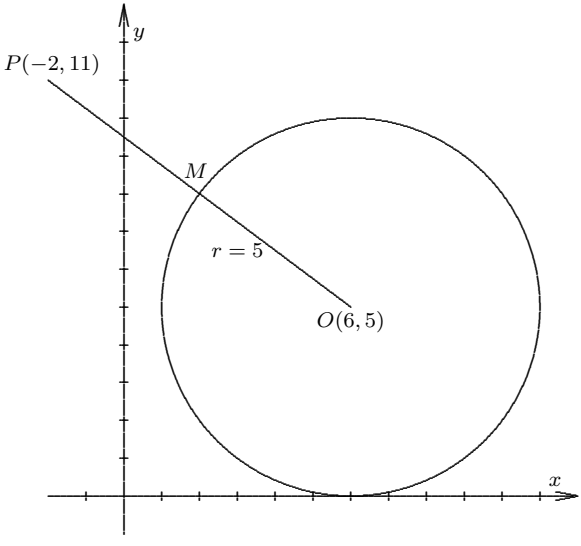
$$S_{k+1} = k^2 + (2k+1) = (k+1)^2$$

2. $n = 3a + 2 = 4b + 3 = 5c + 4 = 6d + 5 = 7e + 6$. Then $n + 1$ is divisible by 3, 4, 5, 6, 7 (since, for example, $n + 1 = 4b + 4 = 4(b + 1)$). The smallest such positive $n + 1$ is lcm (least common multiple) of 3, 4, 5, 6, 7, that is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. Therefore, the smallest such positive n is 419.

3. Draw radii OE, OF perpendicular to sides AB, BC, respectively. Then quadrilateral $BEOF$ is a square. Furthermore, triangle $COF \sim$ (similar) triangle CBA so $\frac{15-r}{r} = \frac{15}{20}$ and $300 - 20r = 15r$, $35r = 300$, $r = \frac{60}{7}$.



4. The closest point, M, is on line OP. Since $OP = \sqrt{8^2 + 6^2} = 10$, and $r = 5$, $OM = \frac{1}{2}OP$, so M is the midpoint of OP and M is the point $(2, 8)$.



5. $\sqrt{10 + \sqrt{84}} = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Square both sides to get $10 + 2\sqrt{21} = a + b + 2\sqrt{ab}$.
Equate rational and irrational parts to get $a + b = 10$ and $ab = 21$. Solve to obtain $(a, b) = (7, 3)$ or $(3, 7)$. Since $a < b$, $(a, b) = (3, 7)$.

6. Let k_1 be the slope of the line $2y + x + 2 = 0$, and k_2 the slope of the line $3y + ax + 2 = 0$. For perpendicularity, $k_1 k_2 = -1$. Therefore $(-\frac{1}{2})(-\frac{a}{3}) = -1$, $a = -6$.

7. We have $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$. The equation with roots $ar + b$ and $as + b$ is

$(x-(ar+b))(x-(as+b))=0$, $x^2-bx+ac=0$.

8. The minimum number of sides in a polygon is 3, so the smallest possible value for the angle x is 60^o. The next largest possible value for x is 90^o and belongs to a four-sided polygon. Each angle of a (convex) polygon is less than 180^o, so $kx<180^o$. Since k is an integer greater than 1, the pair $x=60^o$, $k=2$ furnishes a solution: $x=60^o$, $kx=120^o<180^o$; P_1 is a triangle, P_2 a hexagon.

But if $x>60^o$ or if $k>2$, $kx\geq 180^o$; so there are no other solutions.

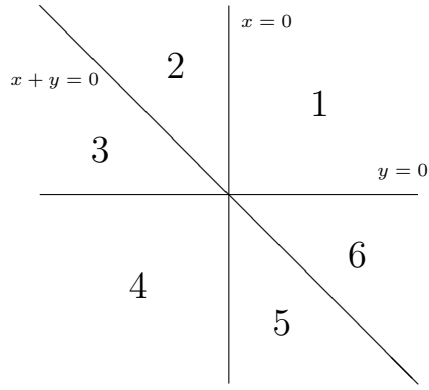
9. There are 8 cases to consider, depending on whether each of x, y and $x+y$ are ≥ 0 or <0.

This is immediately reduced to 6 since

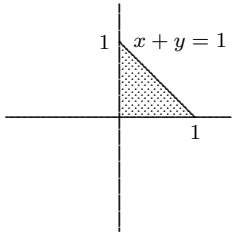
$x\geq 0,\ y\geq 0 \Rightarrow x+y\geq 0$,

and $x<0,\ y<0 \Rightarrow x+y<0$.

These are 6 regions marked

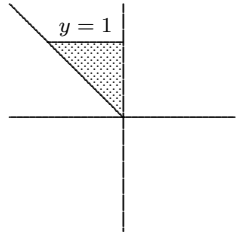


1.



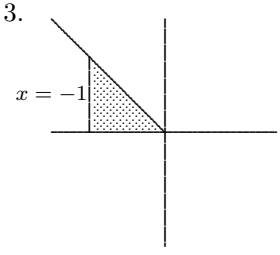
$x\geq 0,\ y\geq 0$.
Here $x+y+x+y\leq 2$,
i.e. $x+y\leq 1$.

2.



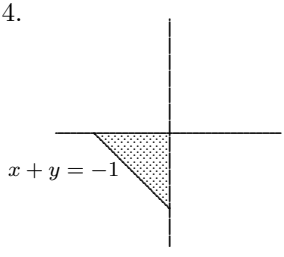
$x<0,\ y\geq 0,\ x+y\geq 0$.
Here $-x+y+x+y\leq 2$,
i.e. $y\leq 1$.

3.



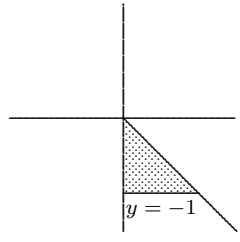
$x<0,\ y\geq 0,\ x+y<0$.
Here $-x+y-x-y\leq 2$,
i.e. $x\geq -1$.

4.



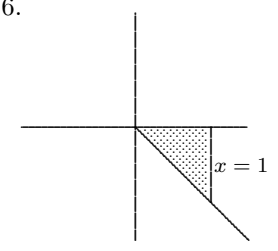
$x<0,\ y<0$,
Here $-x-y-x-y\leq 2$,
i.e. $x+y\geq -1$.

5.



$x\geq 0,\ y<0,\ x+y<0$.
Here $x-y-x-y\leq 2$,
i.e. $y\geq -1$.

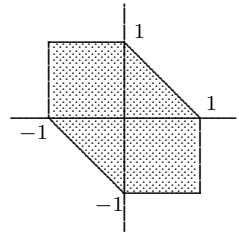
6.



$x\geq 0,\ y<0,\ x+y\geq 0$.
Here $x-y+x+y\leq 2$,
i.e. $x\leq 1$.

Each of these regions is a triangle with area $\frac{1}{2}$.

Therefore total area is equal 3 and looks like:



Rešitve logičnih nalog

1.

Novinarji	Kraji	Zgodbe	Osebe v zgodbah
Ana	Atene	rop	okrožni sodnik
Edi	Dunaj	umor	župnik
Hugo	Amsterdam	osramočenje	mornar
Peter	Rim	pijan voznik	knezova hčer
Sandra	Pariz	škandal v bolnišnici	pevka

2.

Jame	Imena	Vzdevki	Specializacije
1	Agata	Senčna	zdravila
2	Hilda	Grdogleda	napoj za srečo
3	Hilarija	Zlobna	strupi
4	Gizela	Volčja	prekletstvo
5	Regina	Temačna	klicanje duhov

3.

Antikvariat	Ptuj	mošnja	Martin Krpan
Podstrešje	Velenje	čaša	Marija Terezija
Zastavljalnica	Novo mesto	čevelj	Primož Trubar
Stari časi	Celje	prstan	Turj. Rozamunda
Spomini	Maribor	obesek	kralj Matjaž

4.

Ana	tajnica	zamudila letalo	stresla sol
Karla	voznica tovornjaka	izpahnila gleženj	čudne sanje
Janez	voznik avtobusa	izgubil uro	kolcanje
Robert	prodajalec zelenjave	oropan	videl črno mačko
Samo	blagajnik	izgubil službo	razbil zrcalo

Naloge in rešitve pripravil
Tomi Soklič

Avstralska varianta eksternega preverjanja znanja

V okviru štirinajste poletne šole mladih matematikov, ki jo je za najboljših štirinajst osmošolcev z državnega tekmovanja za zlato Vegovo priznanje organiziralo *Društvo matematikov, fizikov in astronomov Slovenije* na Bledu, so udeleženci reševali tudi avstralsko varianto eksternega preverjanja znanja. Naloge so reševali v angleščini, priložen so imeli le slovarček izrazov, za katere smo menili, da jih osmošolci še niso vajeni.

Največ težav so učencem povzročale 16., 22., 24., 25., 27., 28. in 30. naloga.

Ko sem udeležence povprašal, ali jim je več težav povzročal jezik ali naloge same, so skoraj vsi odgovorili, da so bile avstralske naloge zahtevnejše od naših in da posebnih težav zaradi jezika niso imeli, ker so naloge izbirnega tipa, torej so zastavljene nekoliko drugače kot pri nas.

Naloge objavljamo prevedene, da jih bodo lahko ocenili tudi učenci, ki jim angleščina dela preglavice:

1. $0.3 \cdot 2$ je enako:

(**A**) 0.15 (**B**) 0.06 (**C**) 0.6 (**Č**) 0.32 (**D**) 0.9

2. $(0.01)^2$ je enako:

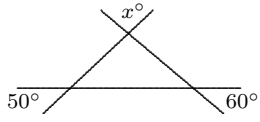
(**A**) 0.1 (**B**) 0.01 (**C**) 0.001 (**Č**) 0.0001 (**D**) 0.0003

3. Vrednost ulomka $\frac{0.75}{15}$ je:

(**A**) 5 (**B**) 0.5 (**C**) 0.05 (**Č**) 0.005 (**D**) 0.0005

4. Koliko stopinj meri kot x na sliki:

(**A**) 50 (**B**) 60 (**C**) 70 (**Č**) 110 (**D**) 65



5. Aritmetična sredina števil 0.1, 0.11 in 0.111 je:

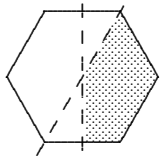
(**A**) 0.041 (**B**) 0.107 (**C**) 0.11 (**Č**) 0.1111 (**D**) 0.17

6. Naj bo $a = 0,6$, $b = 1,2$ in $c = 0,4$. Potem je vrednost ulomka $\frac{ab}{c}$ enaka:

(**A**) 1.8 (**B**) 18 (**C**) 0.18 (**Č**) 0.018 (**D**) 180

7. Obe črtkano narisani sečnici v tem pravilnem šestkotniku sta njegovi somernici. Kolikšen del ploščine šestkotnika predstavlja osenčeni del?

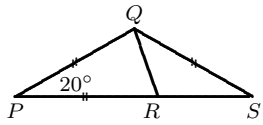
(**A**) $\frac{5}{12}$ (**B**) $\frac{7}{24}$ (**C**) $\frac{11}{24}$ (**Č**) $\frac{1}{3}$ (**D**) $\frac{3}{8}$



8. Katero od naštetih števil je najmanjše?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{2}{7}$ **(Č)** $\frac{3}{10}$ **(D)** $\frac{3}{11}$

9. Na sliki je $\overline{PQ} = \overline{PR} = \overline{QS}$ in $\sphericalangle QPR = 20°$. Koliko stopinj meri kot $\sphericalangle RQS$?



(A) 20 **(B)** 40 **(C)** 60 **(Č)** 80 **(D)** 100

10. Moji otroci so stari šest, osem in deset let. Skupaj dobijo 12 dolarjev žepnine tedensko, razdelijo pa jo premosorazmerno s svojo starostjo. Koliko dolarjev dobi najstareši?

(A) 3 **(B)** 4 **(C)** 5 **(Č)** 6 **(D)** 10

11. Sliki dveh ulomkov sta na številski premici enako oddaljeni od $\frac{1}{4}$ in $\frac{2}{3}$ ter med seboj. Manjši ulomek je:

(A) $\frac{13}{24}$ **(B)** $\frac{7}{18}$ **(C)** $\frac{29}{36}$ **(Č)** $\frac{5}{12}$ **(D)** $\frac{1}{3}$

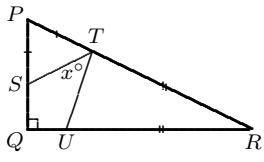
12. Razlika kvadratov dveh zaporednih naravnih števil je d. Manjše število lahko izrazimo kot:

(A) $d-1$ **(B)** $\frac{d-1}{2}$ **(C)** $\frac{d+1}{2}$ **(Č)** $\frac{d}{2}$ **(D)** $(d-1)^2$

13. Normalni racman ima dve nogi. Šepavi racman ima eno nogo. Sedeči racman nima nog. Imamo 33 racmanov s skupno 32 nogami. Skupno število normalnih in šepavih racmanov je dvakrat večje od števila sedečih racmanov. Šepavih racmanov je:

(A) 9 **(B)** 10 **(C)** 11 **(Č)** 12 **(D)** 13

14. Trikotnik $\triangle PQR$ je pravokoten s pravim kotom v oglišču Q. Na njegovih stranicah so izbrane točke S, T in U, tako da velja $\overline{PS} = \overline{PT}$ in $\overline{RT} = \overline{RU}$. Koliko stopinj meri kot $\sphericalangle STU$?



(A) 30 **(B)** 45 **(C)** 50 **(Č)** 55 **(D)** 60

15. Okroglo ploščo želimo popolnoma prekriti s samimi okroglimi ploščami, ki so enako velike kot prekrita plošča. Če gledamo z vrha, se prekrivne plošče središča prekrite plošče lahko dotikajo, ne smejo pa ga pokriti. Najmanj koliko plošč potrebujemo za pokrivanje?

(A) 6 **(B)** 4 **(C)** 3 **(Č)** 2 **(D)** 5

16. Zidar potrebuje za gradnjo 10000 zidakov. Iz izkušenj ve, da se pri transportu razbije največ 7 % zidakov. Zidake prodajajo v svežnjih po 100. Najmanj koliko zidakov mora naročiti, da jih bo imel zagotovo dovolj?

(A) 10900 **(B)** 10600 **(C)** 10500 **(Č)** 10700 **(D)** 10800

17. Slika prikazuje tabelo 5×5. V zgornji vrsti so simboli P, Q, R, S in T. V četrti vrsti so na sredini simboli P, Q in R. V ostala polja lahko vpišemo simbole P, Q, R, S in T tako, da nobena vrsta, noben stolpec in nobena diagonala ne vsebuje istega simbola več kot enkrat. Kateri simbol bo v osenčenem polju?

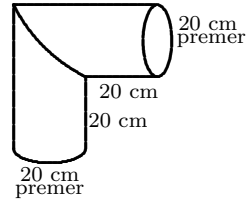
P	Q	R	S	T
	P	Q	R	

(A) P **(B)** Q **(C)** R **(Č)** S **(D)** T

18. Na piknik plišastih medvedkov je vsak otrok prinesel dva medvedka. Drugih medvedkov na prireditvenem prostoru ni bilo. Ob koncu so ugotovili, da je 32 otrok izgubilo enega ali oba svoja medvedka, 56 medvedkov brez lastnika pa je ležalo po tleh. Koliko otrok je izgubilo natanko enega medvedka?

(A) 8 **(B)** 12 **(C)** 16 **(Č)** 20 **(D)** 24

19. Koleno dimniške cevi je izdelano iz pločevine kot kaže slika. Premer cevi je 20 cm, ostale podatke pa dobiš na sliki. Koliko cm^2 pločevine je potrebno za izdelavo takega kolena?



(A) 1200π **(B)** 800π **(C)** 6000π **(Č)** 4000π **(D)** 600π

20. Naj bo $a^2 = a + 2$. Tedaj je a^3 enako:

(A) $a + 4$ **(B)** $2a + 8$ **(C)** $3a + 2$ **(Č)** $4a + 8$ **(D)** $27a + 8$

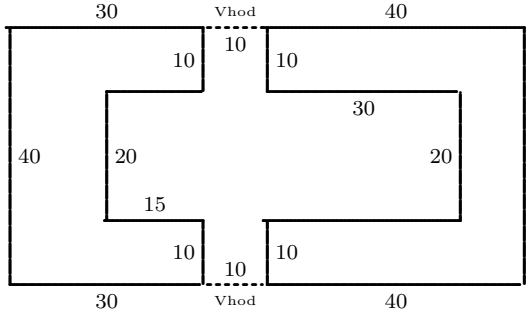
21. Talna ploščica ima obliko pravilnega večkotnika. Če jo odlepimo in zavrtimo za $50°$, jo lahko ponovno namestimo na svoje mesto na tleh. Najmanjše število stranic, ki jih ima ta večkotnik, je:

(A) 8 **(B)** 24 **(C)** 25 **(Č)** 30 **(D)** 36

22. Trajekt prepluje razdaljo med krajema A in B v 33 minutah, medtem ko hidrogliser potrebuje za isto razdaljo le 15 minut. Nekega dne je trajekt odplul iz kraja A ob 11^{53}, hidrogliser pa ob 12^{05}. Hidrogliser je trajekt prehitel ob:

(A) 12^{11} **(B)** 12^{12} **(C)** 12^{13} **(Č)** 12^{14} **(D)** 12^{15}

23. Na sliki je načrt stavbe z dvoriščem in dvojimi vhodnimi vrati. Mimoidoči lahko gledajo skozi vhodna vrata, ne smejo pa vstopiti. Razsežnosti stavbe so na sliki vpisane v metrih, vsi koti so pravi. Koliko kvadratnih metrov dvorišča mimoidoči ne morejo videti?



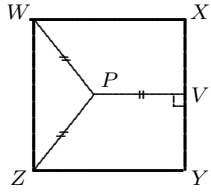
(A) 250 **(B)** 200 **(C)** 300 **(Č)** 400 **(D)** 325

24. Naj bo število $n = 33...3$ zapisano s 100 števkami 3. Število N je zapisano z x števkami 4 in je deljivo z n. Tedaj je x enako:

(A) 180 **(B)** 240 **(C)** 150 **(Č)** 400 **(D)** 300

25. Na sliki je kvadrat $WXYZ$, kjer je $PV \perp XY$. Naj bo $\overline{PW} = \overline{PZ} = \overline{PV} = 10$ cm. Koliko cm^2 meri ploščina kvadrata $WXYZ$?

(A) 225 **(B)** 232 **(C)** 248 **(Č)** 256 **(D)** 324



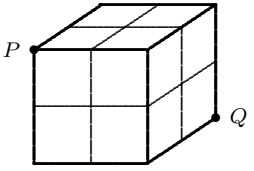
26. Z določenim tipom avtomobilskih gum na prvih kolesih lahko prevozim $40000\ \mathrm{km}$, na zadnjih kolesih pa $60000\ \mathrm{km}$. Največ koliko km lahko prevozim s štirimi gumami, če vmes zamenjam gume s prednjih na zadnji kolesi in obratno?

(A) 52000 **(B)** 50000 **(C)** 48000 **(Č)** 40000 **(D)** 44000

27. V končnem zaporedju petih členov S_0 vsak člen nadomestimo s številom ponovitev tega člena v zaporedju in tako dobimo zaporedje S_1. Če je npr. $S_0 = (1, 2, 3, 1, 2)$, potem je $S_1 = (2, 2, 1, 2, 2)$. Naj bo sedaj S_0 poljubno tako zaporedje. Katero od naštetih zaporedij je lahko S_1?

(A) $(1, 1, 2, 2, 2)$ **(B)** $(1, 1, 1, 2, 2)$ **(C)** $(1, 1, 2, 2, 3)$ **(Č)** $(1, 3, 3, 3, 3)$
(D) $(2, 2, 2, 3, 3)$

28. Vsaka mejna ploskev kocke je razdeljena na štiri enake kvadrate, kot je prikazano na sliki. Iz oglišča P lahko potuješ po površini kocke le po narisanih daljicah, tako da se vsakič za eno približaš oglišču Q. Kolikšno je število možnih poti iz P v Q?



(**A**) 46 (**B**) 90 (**C**) 36 (**Č**) 54 (**D**) 60

29. Števila p, q, r, s in t so zaporedna naravna števila, urejena v naraščajočem zaporedju. Kolikšna je najmanjša vrednost števila r, če je $p+q+r+s+t$ popoln kub in $q+r+s$ popoln kvadrat?

(**A**) 75 (**B**) 288 (**C**) 225 (**Č**) 675 (**D**) 725

30. Na poljubni krožnici izberi 21 točk, ki so med seboj enako oddaljene in jih zaporedno označi z 0, 1, ..., 21. n teh točk pobarvaj rdeče, tako da razdalji med poljubno izbranima paroma rdečih točk nista enaki. Tedaj je n lahko največ:

(**A**) 2 (**B**) 3 (**C**) 4 (**Č**) 5 (**D**) 6

Primerjava rezultatov, ki so jih pri reševanju 30 nalog dosegli učenci:

naloga	pravilni odgovor	odstotek pravilnih rešitev pri nas	v Avstraliji
1	c	100	92
2	d	100	42
3	c	100	52
4	c	100	87
5	b	100	57
6	a	100	39
7	a	100	63
8	a	79	34
9	c	100	30
10	c	100	58
11	b	71	16
12	b	100	23
13	d	86	25
14	b	93	32
15	c	71	19
16	e	43	11
17	b	100	44
18	a	86	43
19	a	100	23
20	c	100	38
21	e	71	9
22	e	21	18
23	e	86	13
24	e	36	7
25	d	29	9
26	c	50	8
27	b	43	14
28	d	29	10
29	d	57	6
30	d	0	21

Glede na to da so pri nas naloge reševali izbrani učenci, rezultatov seveda ne moremo neposredno primerjati z avstralskimi rezultati, ki jih imamo za celotno populacijo, kažejo pa na to, da jezik vendarle ni prehuda ovira pri reševanju matematičnih nalog.

Aleksander Potočnik

Rešitve nalog za najmlajše

2. Peter je smučal, Maja je drsala, Barbara je postavljala snežaka, Tina se je kepala, Rado se je sankal.

3. Marko obiskuje namizni tenis, Peter odbojko, Sonja košarko, Nataša pa strelstvo.

6.

a)

15	13	12
14	20	17
18	16	19

b)

5	3	2
4	10	7
8	6	9

c)

6	4	3
5	11	8
9	7	10

d)

13	8	1	9
14	2	7	10
6	15	4	12
11	16	3	5

æ

International Mathematical Talent Search

Problems Round 13

Problem 1/13. Milo is a student at Mindbender High. After every test, he figures his cumulative average, which he always rounds to the nearest whole percent. (So 85.49 would round down to 85, but 85.50 would round up to 86.) Today he had two tests. First he got 75 in French, which dropped his average by 1 point. Then he got 83 in History, which lowered his average another 2 points. What is his average now?

Problem 2/13. Erin is devising a game and wants to select four denominations out of the available denominations \$ 1, \$ 2, \$ 3, \$ 5, \$ 10, \$ 20, \$ 25, and \$ 50 for the play money. How should he choose them so that every value from \$ 1 to \$ 120 can be obtained by using at most seven bills?

Problem 3/13. For which positive integers d is it possible to color the integers with red and blue so that no two red points are a distance d apart, and no two blue points are a distance 1 apart?

Problem 4/13. Prove that there are infinitely many ordered triples of positive integers (x, y, z) such that $x^3 + y^5 = z^7$.

Problem 5/13. Armed with just a compass — no straightedge — draw two circles that intersect at right angles; that is, construct overlapping circles in the same plane, having perpendicular tangents at the two points where they meet.

Problems Round 14

Problem 1/14. Let a, b, c, d be positive numbers such that $a^2 + b^2 + (a-b)^2 = c^2 + d^2 + (c-d)^2$. Prave that $a^4 + b^4 + (a-b)^4 = c^4 + d^4 + (c-d)^4$.

Problem 2/14. The price tags on three items in a store are as follows:

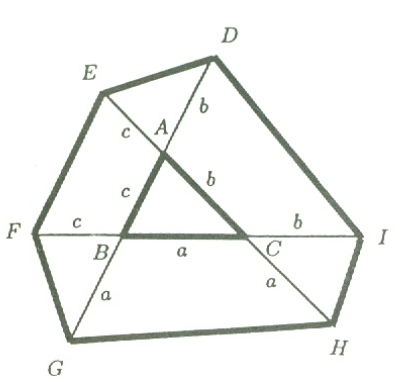
\$ 0.75	\$ 2.00	\$ 5.50

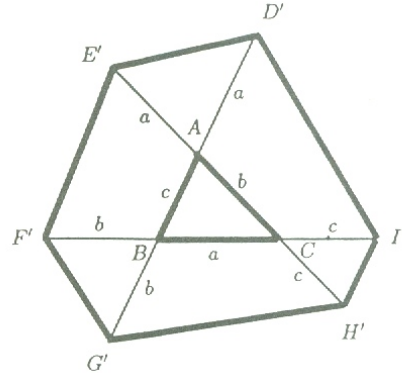
Notice that the sum of these three prices is \$ 8.25, and that the product of these three numbers is also 8.25. Identify four prices whose sum is \$ 8.25 and whose product is also 8.25.

Problem 3/14. In a group of eight mathematicians, each of them finds that there are exactly three others with whom he/she has a common area of interest. Is it possible to pair them off in such a manner that in each of the four pairs, the two mathematicians paired together have no common area of interest?

Problem 4/14. For positive integers a and b, define $a \sim b$ to mean that $ab + 1$ is the square of an integer. Prave that if $a \sim b$, then there exists a positive integer c such that $a \sim c$ and $b \sim c$.

Problem 5/14. Let $\triangle ABC$ be given, extend its sides, and construct two hexagons as shown below. Compare the areas of the hexagons.





Solutions Round 10

Problem 1/10. Find $x^2+y^2+z^2$ if x, y, and z are positive integers such that

$$7x^2-3y^2+4z^2=8 \quad \text{and} \quad 16x^2-7y^2+9z^2=-3.$$

Solution 1: Refer to $7x^2-3y^2+4z^2=8$ as (1) and $16x^2-7y^2+9z^2=-3$ as (2). Then $16\cdot(1)-7\cdot(2)$ produces the equation $y^2+z^2=149$, while $7\cdot(1)-3\cdot(2)$ produces the equation $x^2+z^2=65$. In the set of positive integers, the first of these yields the solutions $(y,z)=(10,7),(7,10)$, while the second one yields $(x,z)=(1,8),(8,1),(7,4),(4,7)$. Comparing these, we find that $z=7$ and that $x=4$ and $y=10$. Upon checking that these values indeed satisfy the original equations, we can compute that $x^2+y^2+z^2=165$.

Solution 2: The system of equations,

$$7x^2-3y^2+4z^2=8, \quad 16x^2-7y^2+9z^2=-3, \quad x^2+y^2+z^2=S,$$

is linear in x^2, y^2, z^2. Upon solving it (using Gaussian elimination or some other method), we find that $x^2=S-149, y^2=S-65, z^2=214-S$. From these, since x^2, y^2, z^2 are positive, it follows that $149<S<214$. In this range, $S-65$ is a perfect square only for $S=165, 186, 209$. For these values we get $214-S=49, 28, 5$. The only one of these producing a perfect square is $S=165$. Since $165-149$ is also a perfect square it follows that the desired answer is indeed 165.

Comments: When we have two equations in three unknowns, we know that in general, we will not find a unique solution based on the equations alone. It is only the further condition that x, y and z are positive integers, that allows us to produce a unique solution.

Alternately, one finds that $9\cdot(1)-4\cdot(2)$ produces $y^2-z^2=84$. This equation may also be solved in integers by factoring both sides, but the previous solutions are more straightforward.

Problem 2/10. Deduce from the simple estimate, $1<\sqrt{3}<2$, that $6<3^{\sqrt{3}}<7$.

Solution 1: $\sqrt{3}<2 \Rightarrow \sqrt{3}+1<3 \Rightarrow 2^{\sqrt{3}+1}<2^3=8<9=3^2 \Rightarrow 2<(3^2)^{1/(\sqrt{3}+1)}=3^{\sqrt{3}-1} \Rightarrow 6<3^{\sqrt{3}} \Rightarrow 18<3^{\sqrt{3}+1} \Rightarrow 4^2<3^{\sqrt{3}+1} \Rightarrow 4^{\sqrt{3}-1}=4^{2/(\sqrt{3}+1)}<3 \Rightarrow 4^{\sqrt{3}}<12<\frac{49}{4} \Rightarrow 4^{\sqrt{3}+1}<7^2 \Rightarrow 4<7^{2/(\sqrt{3}+1)}=7^{\sqrt{3}-1} \Rightarrow 28<7^{\sqrt{3}} \Rightarrow 27<7^{\sqrt{3}} \Rightarrow 3^{\sqrt{3}}<7$.

Solution 2: $1 < \sqrt{3} < 2 \Rightarrow 0 < \sqrt{3} - 1 < 1 \Rightarrow 0 < (\sqrt{3} - 1)^2 < 1 \Rightarrow 0 < \sqrt{3} - (3/2) < 1/2 \Rightarrow 0 < (\sqrt{3} - (3/2))^2 < 1/4 \Rightarrow 5/3 < \sqrt{3} < 7/4 \Rightarrow 3^{5/3} < 3^{\sqrt{3}} < 3^{7/4} \Rightarrow 6 < 3^{\sqrt{3}} < 7$ since $6^3 = 216 < 243 = 3^5 \Rightarrow 6 < 3^{5/3}$ and $3^7 = 2187 < 2401 = 7^4 \Rightarrow 3^{7/4} < 7$.

Comments: Alternately, $5/3 < \sqrt{3}$ follows from $3(2-\sqrt{3}) < (2+\sqrt{3})(2-\sqrt{3}) = 1$, and $\sqrt{3} < 7/4$ follows from $1 = (2 + /3)(2 - /3) < 4(2 - \sqrt{3})$. One can also observe that $1 < \sqrt{3} < 2 \Rightarrow 1 < 2 \Rightarrow 2/9 < 4/9$ and $1/16 < 2/16 \Rightarrow 25/9 < 27/9$ and $48/16 < 49/16 \Rightarrow 5/3 < \sqrt{3}$ and $\sqrt{3} < 7/4$ upon taking square roots. These same bounds on $\sqrt{3}$ can also be obtained by applying the inequalities $1 < \sqrt{3}$ and $\sqrt{3} < 2$ on the right side of the identity $\sqrt{3} = 1 + \frac{1}{1+\frac{1}{1+\sqrt{3}}}$.

Since neither the hypothesis nor the conclusion involved tight bounds, it would have been a violation of the "spirit of the problem" to obtain and utilize sharper approximations than the ones used above.

This problem, as well as the next one, were originally posed by Mr. F. David Hammer some years ago. Mr. Hammer is presently an industrial mathematician/computer analyst; some years ago, during his academic career, he prepared the problems for the excellent Stockton State Mathematics Contest.

Problem 3/10. For each positive integer $n, n \geq 2$, determine a function

$$f_n(x) = a_n + b_n x + c_n |x - d_n|,$$

where a_n, b_n, c_n, d_n depend only on n, such that

$$f_n(k) = k + 1 \quad \text{for} \quad k = 1, 2, \ldots, n-1 \quad \text{and} \quad f_n(n) = 1.$$

Solution 1: For $x \leq d_n, f_n(x) = a_n + c_n d_n + (b_n - c_n)x$, while for $x \geq d_n, f_n(x) = a_n - c_n d_n + (b_n + c_n)x$. Of course, these functions coincide when $x = d_n$. From the fact that for $1 \leq x \leq n-1, f_n(x) = x + 1$, we know that for $x \leq n-1$, the slope of the function, $b_n - c_n = 1$, and the y-intercept, $a_n + c_n d_n = 1$. Since the slope of the function must become negative at some point before $x = n$, in order for $f_n(n) = 1$, we know that $n - 1 \leq d_n < n$. If we choose $d_n = n - 1$, then we have the slope of the second part of the function, $b_n + c_n = 1 - n$, so we can solve for b_n and c_n in terms of n to obtain $b_n = 1 - \frac{n}{2}$ and $c_n = -\frac{n}{2}$, and so $a_n = 1 + \frac{n(n-1)}{2}$.

Solution 2: More generally, recognizing that any d_n in the interval $[n-1, n)$ may be chosen, we find that the slope of the second part of the function is $-\frac{d_n}{n-d_n}$. Now solving for b_n and c_n yields $b_n = 1 - \frac{n}{2(n-d_n)}$ and $c_n = -\frac{n}{2(n-d_n)}$. Thus $a_n = 1 + \frac{nd_n}{2(n-d_n)}$.

Solution 3: One must recognize that the flip-flop behavior of f_n must be caused by d_n; i.e., one must choose d_n so that the sign of $k - d_n$ changes when $k = n$. To this end, d_n must satisfy $n - 1 \leq d_n < n$. Then, for $k = 1, 2, 3, \ldots, n$, we are led to the following equations:

$$a_n + b_n + c_n(d_n - 1) = 2$$
$$a_n + 2b_n + c_n(d_n - 2) = 3$$
$$\vdots$$
$$a_n + (n-1)b_n + c_n(d_n - (n-1)) = n,$$
$$a_n + nb_n + c_n(n - d_n) = 1.$$

From the first two of these we find that $c_n = b_n - 1$, while the first and last equations — upon substituting for c_n — yield $a_n = (n/2) + 1 - nb_n$. Hence, upon substituting these values for a_n and c_n, from the first equation we find that $b_n = 1 - \frac{n}{2(n-d_n)}$, which in turn leads to $a_n = 1 + \frac{nd_n}{2(n-d_n)}$ and $c_n = -\frac{n}{2(n-d_n)}$ where $d_n = n - \varepsilon$ with $0 < \varepsilon \leq 1$. For example, if $d_n = n - (1/2)$, then $(a_n, b_n, c_n, d_n) = ((2n^2 - n + 2)/2, 1 - n, -n, n - (1/2))$.

Problem 4/10. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and

(i) discard them if they are of the same color,
(ii) discard the black ball and return to the bag the red ball if they are different colors.

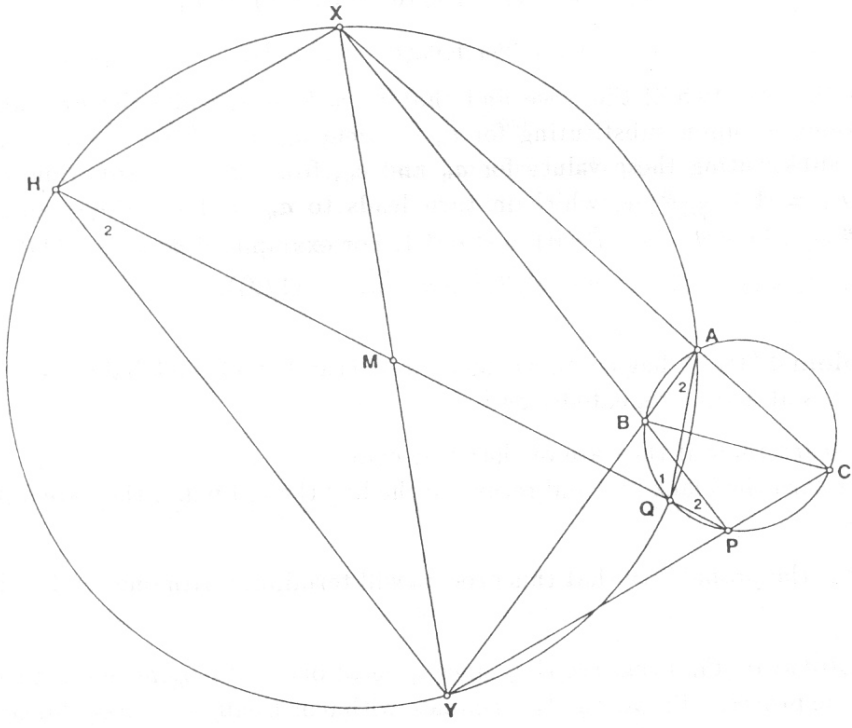
What is the probability that this process will terminate with one red ball in the bag?

Solution: First observe that since at least one ball is removed during each turn, the process will eventually terminate with 0 or 1 ball in the bag. Since black balls can be removed 1 or 2 at a time, the number of black balls will eventually be zero. However, since red balls are removed 2 at a time and since we start with an odd number of red balls, the number of red balls in the bag at any given time will be odd. Therefore, the probability that the process terminates with 1 red ball remaining in the bag is 1.

Comments: This problem was created by Daniel Macks, a former US-AMTS participant. Dan is presently an undergraduate at Brown University.

Problem 5/10. Let P be a point on the circumcircle of $\triangle ABC$, distinct from A, B, and C. Suppose BP meets AC at X, and CP meets AB at Y. Let Q be the point of intersection of the circumcircles of $\triangle ABC$ and $\triangle AXY$, with $Q \neq A$. Prove that PQ bisects the segment XY. (The various points of intersection may occur on the extensions of the segments.)

Solution: We shall complete the proof in the case of an arrangement of points A, B, C, and P on the circumcircle of $\triangle ABC$ satisfying the conditions of the problem and such that both points X and Y lie outside of that circle. Such an arrangement is shown in the figure below; the case where one of X or Y lies inside the circumcircle of $\triangle ABC$ may be treated in a similar fashion.



Referring to the figure above, note that PQ has been extended to meet the circumcircle of $\triangle ABC$ at point H. The reader should recall that the measure of an angle inscribed in a circle equals one half the measure of its intercepted arc. From this result, it follows that two angles inscribed in a circle and intercepting a common arc are equal and also that, in a cyclic quadrilateral, opposite angles are supplementary. We shall be applying these results to four cyclic quadrilaterals of our figure: $ACPQ$ and $ABQP$ (both inscribed in the circumcircle of $\triangle ABC$) and $AQHX$ and $HAQY$ (both inscribed in the circumcircle of $\triangle AXY$).

Since $\angle ACP$ and $\angle AQP$ are opposite angles of $ACPQ$, they are supplementary. So

$$\angle ACP \quad \text{equals} \quad \angle AQH. \tag{1}$$

Since $\angle AQH$ and $\angle HXA$ are opposite angles of $AQHX$,

$$\angle AQH \quad \text{and} \quad \angle HXA \quad \text{are supplementary.} \tag{2}$$

From (1) and (2), it follows that $\angle ACP$ and $\angle HXA$ are supplementary. Thus segments HX and YC must be parallel.

Now $\angle BAQ$ and $\angle BPQ$ are both inscribed in the circumcircle of $\triangle ABC$ and both intercept arc BQ. So $\angle BAQ$ equals $\angle BPQ$. A similar argument in the circumcircle of $\triangle AXY$ reveals that $\angle YAQ$ equals $\angle YHQ$. Since $\angle YAQ$ coincides with $\angle BAQ$, $\angle BPQ$ equals $\angle YHQ$. Thus segments XP and HY must be parallel.

It follows that $XHYP$ is a parallelogram, and its diagonals must bisect each other. Thus, XY is bisected by PH, which is the extension PQ.

Comments: This problem is a generalization of Problem 17055 in the "Mathematical Questions and Solutions" of the *Educational Times* [21(1912):41]. It was proposed by Dr. Stanley Rabinowitz, whose *Index to Mathematical Problems*, 1980–1984 is highly recommended to all problemists. The solution given above (as well as the figure) are also due to him.

Solutions Round 11

Problem 1/11. Express $\frac{19}{94}$ in the form $\frac{1}{m}+\frac{1}{n}$, where m and n are positive integers.

Solution 1: First observe that since $\frac{19}{94} < \frac{1}{4}$, both m and n are greater than 4. To obtain an upper bound on them, note that $\frac{19}{94} = \frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn}$; thus $94(m+n) = 19mn$. Since 47 is a prime divisor of 94, it follows that either m or n is divisible by 47. Without loss of generality, assume that 47 divides n. Thus $\frac{1}{n} \leq \frac{1}{47}$, and $\frac{1}{m} = \frac{19}{94} - \frac{1}{n} > \frac{19}{94} - \frac{1}{47} = \frac{17}{94} > \frac{1}{6}$. It follows that $4 < m < 6$, and hence m can only be 5. When $m = 5$, we see that $\frac{1}{n} = \frac{19}{94} - \frac{1}{5} = \frac{1}{470}$, and the unique answer is

$$\frac{19}{94} = \frac{1}{5} + \frac{1}{470}.$$

Solution 2: The equation $\frac{19}{94} = \frac{m+n}{mn}$ can be written in the form $(19m - 94)(19n - 94) = 94^2$. Let $a = 19m - 94$ and $b = 19n - 94$. Then $m = \frac{a+94}{19}, n = \frac{b+94}{19}$, and it follows that $a + 94 \equiv 0 \pmod{19}$ and $b + 94 \equiv 0 \pmod{19}$. Thus $a \equiv 1 \pmod{19}$ and $b \equiv 1 \pmod{19}$. Since the only positive integers congruent to 1 modulo 19 and less than 94 are $1, 20, 39, 58$ and 77, and since of these only 1 divides 94^2, it follows that either a or b must be 1. With $a = 1$ we obtain $(m, n) = (5, 470)$ as the unique solution.

Solution 3: From the equivalent equation, $(19m - 94)(19n - 94) = 94^2 = 8836$, we find that $19m - 94$ must be a factor of 8836. Since the factors of 8836 are $F = 1, 2, 4, 47, 94, 188, 2209, 4418, 8836$, it follows that $m = \frac{F+94}{19}$. Without loss of generality, $F \leq 94$. We try all possible values of F and get the following possible values of m:

F:	1	2	4	47	94
m:	5	$\frac{96}{19}$	$\frac{98}{19}$	$\frac{141}{19}$	$\frac{188}{19}$

Since m must be an integer, it follows that $m = 5$, which yields $n = 470$, as before.

Solution 4: Assume, without loss of generality, that $m < n$, and hence that $\frac{1}{m} > \frac{1}{n}$. Upon observing that $\frac{1}{5} < \frac{19}{94} < \frac{1}{4}$ and that $\frac{1}{m} < \frac{19}{94} = \frac{1}{m} + \frac{1}{n} < \frac{2}{m}$, it follows that $\frac{1}{m} < \frac{1}{4}$ and $\frac{2}{m} > \frac{1}{5}$. Thus the only possible values for m are: $5, 6, 7, 8, 9$. Of these, only $m = 5$ yields an integer value, 470, for n.

Solution 5: Again, assume that $\frac{1}{m} > \frac{1}{n}$. Since $\frac{1}{5} < \frac{19}{94} < \frac{1}{4}$, it follows that $5 \leq m$. Thus $\frac{19}{94} - \frac{1}{m} \geq \frac{19}{94} - \frac{1}{5} = \frac{1}{470}$. Hence, $5 \leq m < n \leq 470$. We may then do a computer search in this range, and find that $m = 5, n = 470$ is the unique solution.

Comments: More generally, the problem may be recognized as a special case of the identity $\frac{m}{mk-1} = \frac{1}{k} + \frac{1}{k(mk-1)}$, with $m = 19$ and $k = 5$.

Even more generally, the problem belongs to the topic of "Egyptian Fractions". It can be proved that for given a and b, there are positive integers m and n such that $\frac{a}{b} = \frac{1}{m} + \frac{1}{n}$ if and only if b has distinct relatively prime divisors s and t for which $s + t \equiv \pmod{a}$.

One procedure for expressing $\frac{a}{b}$ as the sum of "unit fractions" (i.e., fractions of the form $\frac{1}{n}$) is to let $\lceil b/a \rceil$, the smallest integer greater than b/a (This is the "ceiling function" of b/a). Then $n_1 - b/a < 1$, and hence $an_1 - b < a$. Moreover, $\frac{a}{b} - \frac{1}{n_1} = \frac{an_1 - b}{bn_1}$. Repeat the process with $\frac{an_1 - b}{bn_1}$ in place of $\frac{a}{b}$. Since $an_1 - b < a$, this process must terminate in a finite number, say k steps. When $an_k - b = 1$ we are done. This process is known as the "Greedy Algorithm" and the proof that it terminates relies on the "Well-Ordering Property" of the natural numbers: *Any nonempty subset of the natural numbers has a smallest element.*

Problem 2/11. Let n be a positive integer greater than 5. Show that at most eight members of the set $\{n + 1, n + 2, \ldots, n + 30\}$ can be primes.

Solution 1: Observe that all members of the set $S_n = \{n + 1, n + 2, \ldots, n + 30\}$ can be expressed in the form $30N + i$, where N is a non-negative integer (which may not be the same for all members of S_n) and i ranges from 0 to 29. Note that $30N + i$ is a multiple of 2 for $i = 0, 2, 4, \ldots, 28$; it is a multiple of 3 for $i = 0, 3, 6, \ldots, 27$; and it is a multiple of 5 for $i = 0, 5, 10, \ldots, 25$. Hence, for $n > 5$, the only values of i for which $30N + i$ could be prime are 1, 7, 11, 13, 17, 19, 23, and 29. Therefore, for $n > 5$, there can be at most eight primes in S_n

Solution 2: By the Inclusion - Exclusion Principle, there are

$$\left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{3} \right\rfloor + \left\lfloor \frac{30}{5} \right\rfloor - \left\lfloor \frac{30}{2 \cdot 3} \right\rfloor - \left\lfloor \frac{30}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{30}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{30}{2 \cdot 3 \cdot 5} \right\rfloor = 22$$

multiples of 2, 3, and 5 among any 30 consecutive positive integers. Hence if 2, 3, and 5 are not in the set $S_n = \{n + 1, n + 2, \ldots, n + 30\}$, then there are at least 22 composite numbers in S_n. Therefore, for $n > 5$, there are at most $30 - 22 = 8$ primes in S_n.

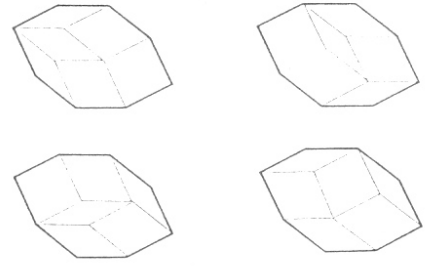
Solution 3: For $n > 5, 15$ of the 30 members of $S_n = \{n+1, n+2, \ldots, n+30\}$ are multiples of 2. Of the remaining 15 members of S_n, 5 are multiples of 3. And of the remaining 10 numbers, 2 must be multiples of 5. Therefore $15 + 15 + 2$ members of S_n must be multiples of 2,3, or 5, and hence can't be primes. That leaves at most $30 - 22 = 8$ primes in S_n.

Solution 4: Let $S_n = \{n+1, n+2, \ldots, n+30\}$, and observe that for each $d \in \{1, 3, 7, 9\}$ there must be three members of S_n whose last digit is d. When $n > 5$, these are the only candidates for primes since all numbers ending in 5 are multiples of 5, and those ending in an even digit are multiples of 2. Next observe that for each $k = 0, 1, 2, \ldots$, one of the three numbers, $k, k+10$, or $k+20$ must be a multiple of 3. Hence, for each $d \in \{1, 3, 7, 9\}$, at most two the members of S_n which end in d can be primes. Therefore, S_n can contain most $4 \cdot 2 = 8$ primes.

Solution 5: For $k > 5$, let $S_k = \{k+1, k+2, \ldots, k+30\}$, and let $s(k)$ be the statement "there are 22 multiples of 2, 3, or 5 in S_k." After verifying that $s(6)$ is true, assume that $s(k)$ is true for $k = n$. Then if $n+1$ is a multiple of 2, or 5, then so is $n+31 = (n+1)+30$, and hence S_{n+1} must also have 22 multiples of 2, 3, or 5. (We lose one, $n+1$, as we go from S_n to S_{n+1}, but gain one, $n+31$, in its place.) On the other hand, if $n+1$ is not a multiple of 2, 3, or 5, then all 22 multiples of 2, 3, and 5 must be in the subset $\{n+2, n+3, \ldots, n+30\}$ of S_n. Since this subset is also a subset of S_{n+1}, it follows that $s(n+1)$ must be true. This completes the induction, and it follows that $s(k)$ is true for all $k > 5$.

Comments: It is interesting to note that in addition to S_6, one finds that S_{1273} also contains eight primes: 1277, 1279, 1283, 1289, 1291, 1297, 1301, 1303. Is it true that for every positive integer N there is a positive integer n such that $n > N$ and S_n contains eight primes? The original and the present problem may also be generalized to longer sequences of consecutive integers.

Problem 3/11. A convex $2n$-gon is said to be "rhombic" if all of its sides are of unit length and if its opposite sides are parallel. As exemplified on the right (for the case of $n = 4$), a rhombic $2n$-gon can be dissected into rhombi of sides 1 in several different ways.



For what value of n can a rhombic $2n$-gon be dissected into 666 rhombi?

Solution 1: We will first prove by induction that a convex rhombic $2n$-gon can be decomposed into $\binom{n}{2} = \frac{n(n-1)}{2}$ unit rhombi. The base case for $n = 2$ is true, since a rhombic 4-gon is already a single rhombus, and $\binom{2}{2} = \frac{2(2-1)}{2} = 1$. Next we prove that any rhombic $2n$-gon can be decomposed into $n-1$ unit rhombi and a $2(n-1)$-gon. Choose one side of the rhombic $2n$-gon, and orient the $2n$-gon so that the chosen side is at the top. Denote the right-hand endpoint of the chosen side as vertex V_1 and number the successive vertices clockwise. From each vertex, beginning with vertex V_2, draw a unit line segment, parallel to the chosen side, extending into the interior of the $2n$-gon, until vertex V_{n-1} is reached. No line segment can be drawn from vertex V_n, since, by the definition of the rhombic $2n$-gon, the side between vertices V_n and V_{n+1} is parallel to the chosen side. Now complete the $n-1$ unit rhombi along the right side of the $2n$-gon. Since, for each rhombus, the top and bottom sides are of unit length and parallel and the right sides are of unit length, the left sides are parallel to the right sides, and are also of unit length. Thus the left sides are also parallel to the opposite sides of the original $2n$-gon. Thus the un-subdivided polygon remaining is a rhombic $2(n-1)$-gon, and it is convex because we have merely translated its right side one unit to the left. By the induction hypothesis, this rhombic $2(n-1)$-gon can be decomposed into $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ unit rhombi, so the rhombic $2n$-gon can be decomposed into $\frac{(n-1)(n-2)}{2} + n - 1 = \frac{(n-1)(n-2)+2n-2}{2} = \frac{n^2-3n+2+2n-2}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$ unit rhombi. Therefore, the problem is simplified to solving the quadratic equation $\binom{n}{2} = 666$. It can be written as $(n+36)(n-37) = 0$, whose only positive solution is $n = 37$. That's the solution to our problem.

Solution 2: First observe that a rhombic $2n$-gon is completely determined by the sequence of its n interior angles, beginning at a chosen vertex. Such a rhombic $2n$-gon can be constructed using unit rhombi (thus proving that it can be decomposed into unit rhombi) using the following algorithm:

> *Initialization:* Choose a side of the rhombic $2n$-gon and place it at the top, and number the interior angles that determine the rhombic $2n$-gon sequentially (clockwise) from a_1 to a_n, starting at the vertex that is the right endpoint of the chosen side. Number the chosen side s_1 and number the sides sequentially (clockwise) also. Now draw a horizontal unit line segment (call it $\hat{e}_1$ and, at an angle the same measure as angle a_1, draw another unit line segment emanating from the right-hand endpoint of $\hat{e}_1$. Call this new segment $\hat{e}_2$, and complete the rhombus determined by these two sides. Set $i = 2$.
>
> *Loop:* While $i < n$, orient the $2i$-gon so that the edge $\hat{e}_1$ is horizontal at the top, and draw a unit segment from the right-hand endpoint of edge e_i, at an angle having the same measure as angle a_i; call this segment $\hat{e}_{i+1}$. Draw a unit segment parallel to $\hat{e}_{i+1}$ emanating from the left-hand endpoint of each unlabeled edge, and complete the i unit rhombi determined by these segments and the rhombic $2i$-gon. Replace i by $i+1$.

Since each iteration produces i more rhombi, the total number of rhombi produced in constructing a rhombic $2n$-gon will be

$$1 \quad \text{(from the initialization)} + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

As in Solution 1, the equation $\binom{n}{2} = 666$ yields $n = 37$.

Solution 3: Construct an inner 'ring' of rhombi as follows. Complete the rhombus defined two adjacent sides. Now complete all possible rhombi around the perimeter of the $2n$-gon. The result is $1 + (n-1) + (n-3) = 2n-3$ unit rhombi (see Solution 1) and a $(2n-4)$-gon. This, together with the special cases for $n = 2$ and $n = 3$, establishes the recursion. Letting $R(n)$ be the number of rhombi into which a rhombic $2n$-gon can be decomposed, we find that $R(2) = 1, R(3) = 3$, and $R(n) = R(n-2) + 2n - 3$ for $n \geq 4$. This can be solved or simply evaluated until it is determined that $R(37) = 666$.

Comments: While in Solution 1, one first had to guess the number of unit rhombi resulting from the dissection of a rhombic $2n$-gon, Solutions 2 and 3 were constructive. In the problem (as well as in the solutions), we have assumed that the $2n$-gon is convex; was that assumption necessary? What happens if the $2n$-gon is concave? One may also inquire about the number of distinct dissections of a rhombic $2n$-gon; that problem seems to be more difficult.

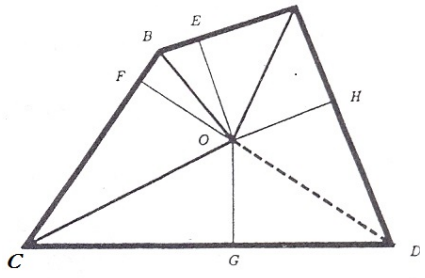
Problem 4/11. Prove that if three of the interior angle bisectors of a quadrilateral intersect at one point, then all four of them must intersect at that point

Solution 1: Let $ABCD$ be a quadrilateral, and assume that

$$\angle OAB = \angle OAD,\ \angle OBA = \angle OBC,\ \text{ and } \angle OCB = \angle OCD,$$

where O is the intersection point of the angle bisectors at A, B, and C. Let E, F, G, and H be the feet of the perpendiculars from O to the sides of the quadrilateral, as shown in the figure below. Then $\triangle AEO$ and $\triangle AOH$ are congruent, since $\angle OAE = \angle OAH$, $\angle OEA = \angle OHA$, and $\overline{OA} = \overline{OA}$. Therefore, $\overline{OA} = \overline{OH}$. One can similarly show that $\overline{OE} = \overline{OF}$ and $\overline{OF} = \overline{OG}$. Therefore, one may conclude that $\overline{OH} = \overline{OG}$.

To conclude the solution, we must still show that $\angle ODG = \angle ODH$. This can be done in several ways. For instance, if we construct a circle with radius $\overline{OG}$ and center O, then since GD and DH are tangent to the circle, the line OD will bisect the angle at D.

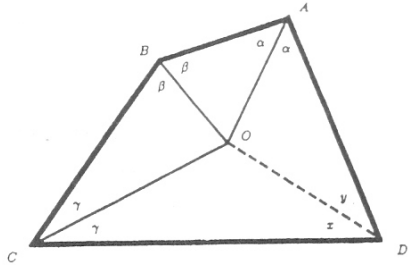


Alternately, one can argue that the right triangles $\triangle OGD$ and $\triangle ODH$ are congruent, as they have the same hypotenuse and equal legs.

Solution 2: Consider the figure shown below, and apply the Law of Sines to $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, and $\triangle DAO$. Then the following equations will result:

$$\frac{\sin\alpha}{\overline{BO}} = \frac{\sin\beta}{\overline{AO}}, \quad \frac{\sin\beta}{\overline{CO}} = \frac{\sin\gamma}{\overline{BO}}, \quad \frac{\sin\gamma}{\overline{DO}} = \frac{\sin x}{\overline{CO}}, \quad \text{and} \quad \frac{\sin y}{\overline{AO}} = \frac{\sin\alpha}{\overline{DO}}.$$

Setting the product of the left hand sides equal to the product of the right hand sides of the above equations produces (after cancellations) the equation $\sin x = \sin y$. Since $x + y \neq 180°$, it follows that $x = y$. Therefore, DO bisects $\angle ADC$, as claimed.



Comments: More generally, one can show that if $n-1$ of the interior angle bisectors of an n-gon intersect in one point, then all n of them must intersect in that point. One may also note that if three of the angle bisectors of a quadrilateral intersect in one point, then one can inscribe a circle in the quadrilateral. Does this property generalize to n-gons?

Problem 5/11. Let $f(x) = x^4 + 17x^3 + 80x^2 + 203x + 125$. Find the polynomial, $g(x)$, of smallest degree for which $f(3 \pm \sqrt{3}) = g(3 \pm \sqrt{3})$ and $f(5 \pm \sqrt{3}) = g(5 \pm \sqrt{5})$.

Solution 1: Let $g(x)$ be any such polynomial and consider the polynomial $h(x) = f(x) - g(x)$. Then $h(3 \pm \sqrt{3}) = 0$ and $h(5 \pm \sqrt{5}) = 0$; i.e., $h(x)$ is divisible by

$$(x - 3 - \sqrt{3})(x - 3 + \sqrt{3})(x - 5 - \sqrt{5})(x - 5 + \sqrt{5})$$

$$= (x^2 - 6x + 6)(x^2 - 10x + 20) = x^4 - 16x^3 + 86x^2 - 180x + 120.$$

Thus, there is a polynomial $a(x)$ such that

$$f(x) - g(x) = h(x) = a(x)(x^4 - 16x^3 + 86x^2 - 180x + 120),$$

and therefore

$$\begin{aligned} g(x) &= f(x) - a(x)(x^4 + 16x^3 + 86x^2 - 180x + 120) \\ &= (x^4 + 17x^3 + 80x^2 + 203x + 125) \\ &- a(x)(x^4 - 16x^3 + 86x^2 - 180x + 120). \end{aligned}$$

Now $g(x)$ will be of degree less than 4 if and only if $a(x) \equiv 1$. In that case $g(x) = 33x^3 - 6x^2 + 383x + 5$, and this is the desired polynomial.

Solution 2: We are seeking the polynomial $g(x)$ of least degree for which the graph passes through the four points $(3 \pm \sqrt{3}, f(3 \pm \sqrt{3})) = (3 \pm \sqrt{3}, 2864 \pm 1337\sqrt{3})$ and $(5 \pm \sqrt{5}, f(5 \pm \sqrt{5})) = (5 \pm \sqrt{3}, 8340 \pm 2963\sqrt{3})$. Such a polynomial of degree less than or equal to 3 exists and, letting it be $g(x) = ax^3 + bx^2 + cx + d$, we need to require that the four linear equations

$$a(3 \pm \sqrt{3})^3 + b(3 \pm \sqrt{3})^2 + c(3 \pm \sqrt{3}) + d = 2864 \pm 1337\sqrt{3}$$

$$a(5 \pm \sqrt{5})^3 + b(5 \pm \sqrt{5})^2 + c(5 \pm \sqrt{5}) + d = 8340 \pm 2963\sqrt{5}$$

be satisfied. This system has the unique solution $a = 33, b = -6, c = 383$, and $d = 5$. Therefore, the desired polynomial is of degree 3 (otherwise we would have $a = 0$, for degree 2, or $a = b = 0$, for degree 1, or $a = b = c = 0$, for degree 0), and it is $g(x) = 33x^3 - 6x^2 + 383x + 5$.

Comments: Clearly, the first solution is the less computational, and hence it is preferable. For the second solution, it is advisable to utilize a computer algebra system (CAS), rather than allowing for possible roundings and hence loss of accuracy.